Separability criterion and inseparable mixed states with positive partial transposition

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Abstract

It is shown that any separable state on Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, can be written as a convex combination of N pure product states with $N \leq (dim\mathcal{H})^2$. Then a new separability criterion for mixed states in terms of range of density matrix is obtained. It is used in construction of inseparable mixed states with *positive* partial transposition in the case of 3×3 and 2×4 systems. The states represent an entanglement which is hidden in a more subtle way than it has been known so far.

I. INTRODUCTION

The problem of quantum inseparability of mixed states has attracted much attention recently and it has been widely considered in different physical contexts (see [1] and references therein). In particular effective criterion of separability of 2×2 and 2×3 systems has been obtained [2,3]. Quite recently the criterion has been used for characterisation of two-bit quantum gate [4] and quantum broadcasting [5]. It also allowed to show that any inseparable state of 2×2 system can be distilled to a singlet form [6].

Recall that the state ϱ acting on the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is called separable if it can be wiritten or approximated (in the trace norm) by the states of the form

$$\varrho = \sum_{i=1}^{k} p_i \varrho_i \otimes \tilde{\varrho}_i \tag{1}$$

where ϱ_i and $\tilde{\varrho}_i$ are states on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Usually one deals with the case $dim\mathcal{H} = m$. For this case it will appear subsequently that any separable state can be written as a convex combination of finite product pure states i.e. in those cases the "approximation" part of the definition appears to be redundant.

Peres has shown [2] that the necessary condition for separability of the state ϱ is positivity of its partial transposition ϱ^{T_2}. The latter associated with an arbitrary product orthonormal $f_i \otimes f_j$ basis is defined by the matrix elements:

$$\varrho^{T_2}_{m\mu,n\nu} \equiv \langle f_m \otimes f_\mu | \varrho^{T_2} | f_n \otimes f_\nu \rangle = \varrho_{m\nu,n\mu}. \tag{2}$$

Although the matrix ϱ^{T_2} depends on the used basis, its eigenvalues do not. Consequently, for any state the condition can be checked using *an arbitrary* product orthonormal basis. [1] It has been shown [3] that for the systems 2×2 and 2×3 the partial transposition condition is also sufficient one. Thus in those cases the set of separable states has been characterized completely in a simple way.

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[1] As the full transposition of positive operator is also positive, the positivity of the partial transposition ϱ^{T_2} is equivalent to positivity of (defined in analogous way) partial transposition ϱ^{T_1}.

For higher dimensions the necessary and sufficient condition for separability has been provided [3] in terms of positive maps. Namely the state ϱ acting on Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is separable iff for any positive map $\Lambda : \mathcal{B}(\mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_1)$ the operator $I \otimes \Lambda\varrho$ is positive ($\mathcal{B}(\mathcal{H}_i)$ denote the set of all operators acting on $\mathcal{H}_i$ and I is identity map). Then a natural question arose, whether the partial transposition condition is also sufficient for higher dimensions. The negative answer to this question has been established [3] with no, however, explicit counterexample given. In this Letter we provide (Sec. 3) a new criterion for separability of quantum states. It is done with the help of analysis of range of density matrices via result on the decomposition of separable states on pure product states (Sec. 2). Namely it appears that any separable states can be written as a convex combination of *finite* number N of pure product states with N restricted by squared dimension of respective Hilbert space. In Sec. 4 we construct families of inseparable states with *positive* partial transposition for 3×3 and 2×4 systems. We achieve our goal using the separability criterion and the technique introduced by Woronowicz in his paper (1976) [9] which has provided heuristic basis for the present analysis. It appears that, in general, the new criterion is rather independent than equivalent to the partial transposition one.

II. FINITE DECOMPOSITION OF SEPARABLE STATES

We prove here the following

Theorem 1 *Let ϱ is a separable state acting on the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, $dim\mathcal{H} = m < \infty$. Then there exists set of N, $N \leq m^2$ product projectors $P_{\psi_i} \otimes Q_{\phi_k}$, $\{i,k\} \in I$ (I is a finite set of pairs of indices with number of pairs $N = \#I \leq m^2$) and probabilities p_{ik} such that*

$$\varrho = \sum_{\{i,k\}\in I} p_{ik} P_{\psi_i} \otimes Q_{\phi_k}. \tag{3}$$

Proof.- The proof depends on properties of compact convex sets in real finitedimensional spaces. Namely then the set of separable states M_{sep} defined above, can be treated as a

compact convex subset of finite-dimensional real space (obtained by real linear combinations of Hermitian operators) with a dimension $n = m^2 - 1$. The set of separable states M_{sep} is a closed convex subset of covex set of all quantum states P which is bounded, compact and given by $n = m^2 - 1$ real parameters in Hilbert-Schmidt basis.

Let us denote by $P_{sep} \subset M_{sep}$ set of all separable pure states. P_{sep} is obviously compact (it is a tensor product of two spheres which are compact in finite dimensional case). We have from the definition of M_{sep} immedately :

$$M_{sep} = \overline{conv P_{sep}}. \tag{4}$$

Here conv A denotes convex hull of A and means set of all possible finite convex combinations of elements from A [2]. $\overline{B}$ stands for the closure of B in the trace norm topology. It appears as the standard fact from the convex set theory that convex hull of any compact set from finite dimensional space is compact itself (see Ref. [7], theorem 14, p. 210). Thus closure of $M_{sep} = \overline{conv P_{sep}} = conv P_{sep}$. Hence set of extremal points of M_{sep} if equivalent to P_{sep}. Then we can apply the Caratheodory theorem [8] which says that any element of compact convex subset of $\mathcal{R}^n$ (in our case $n = m^2 - 1$) can be represented as a convex combination of (at most n+1) afinely independent extreme points [8]. Usage of this theorem completes the proof of our statement.

III. SEPARABILITY CRITERION

First we will need the following

Lemma 1 *Let state ϱ act on the Hilbert space $\mathcal{H}$, $dim\mathcal{H} < \infty$. Then for an arbitrary ϱ–ensemble $\{\Psi_i, p_i\}$:*

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[2] The convex hull of the set of A is usually defined as a minimal convex set containing A, but it is shown that it is equivalent to the set of all possible finite convex combinations of element of A (see Ref. [7]).

$$\varrho = \sum_i p_i |\Psi_i >< \Psi_i| \tag{5}$$

each of vectors Ψ_i belongs to the range of the state ϱ.

Proof.- The range of ϱ is defined by $Ran\varrho \equiv \{\psi \in \mathcal{H}\colon\ \varrho\phi = \psi$ for some $\phi \in \mathcal{H}\}$. As ϱ is linear and hermitian operator we have that $Ran\varrho$ is simply a subspace spanned by all eigenvectors of ϱ belonging to nonzero eigenvalues. In short, $Ran\varrho$ is a support of ϱ. Following [10] we have that Ψ_i belongs to the support of ϱ. Thus any Ψ_i belongs to $Ran\varrho$.

Now we can prove our main result:

Theorem 2 *Let ϱ act on Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, $dim\mathcal{H} = m$. If ϱ is separable then there exists a set of product vectors $\{\psi_i \otimes \phi_k\}$, $\{i,k\} \in I$ (I is a finite set of pairs of indices with number of pairs $N = \#I \leq m^2$) and probabilities p_{ik} such that*

(i) the ensemble $\{\psi_i \otimes \phi_k,\, p_{ik}\ \}$, ($\{\psi_i \otimes \phi_k^\star,\, p_{ik}\ \}$) corresponds to the matrix ϱ, (ϱ^{T_2}),

(ii) the vectors $\{\psi_i \otimes \phi_k\ \}$, ($\{\psi_i \otimes \phi_k^\star\ \}$) span the range of ϱ (ϱ^{T_2}), in particular any of vectors $\{\psi_i \otimes \phi_k\}$ ($\{\psi_i \otimes \phi_k^\star\}$) belongs to the range of ϱ (ϱ^{T_2}).

Proof.- Let us prove first the statement (i). According to the theorem form Section II any separable state ϱ can be written in the form

$$\varrho = \sum_{\{i,k\}\in I} p_{ik} P_{\psi_i} \otimes Q_{\phi_k} \equiv \sum_{\{i,k\}\in I} p_{ik} |\psi_i \otimes \phi_k >< \psi_i \otimes \phi_k|. \tag{6}$$

using only $N = \#I \leq m^2$ pure product states $P_{\psi_i} \otimes Q_{\phi_k}$. Remembering that the transposition of Hermitian operator is simply equivalent to the complex conjugation of its matrix elements we get

$$Q_{\phi_k}^T = Q_{\phi_k}^\star = (|\phi_k >< \phi_k|)^\star = |\phi_k^\star >< \phi_k^\star| = Q_{\phi_k^\star}. \tag{7}$$

From the above and the definition of partial transposition (2) we obtain

$$\varrho^{T_2} = \sum_{\{i,k\}\in I} p_{ik} P_{\psi_i} \otimes Q_{\phi_k}^T \equiv \sum_{\{i,k\}\in I} p_{ik} |\psi_i \otimes \phi_k^\star >< \psi_i \otimes \phi_k^\star|. \tag{8}$$

hence we obtain the statement (i). Obviously, any vector ψ from the range of the state is given by a linear combination of vectors belonging to the ensemble realising the state. Using the lemma immediately completes the proof of (ii).

Remark 1.- Using the full transposition one can easily see that the analogous theorem (with vectors conjugated on the first space) equivalent to the above one is valid for ϱ^{T_1}.

Remark 2.- The conjugation $\phi^\star$ associated with the basis the transposition of Q_ϕ was performed in is simply obtained by complex conjugation of the coefficients in this basis up to the irrelevant phase factor. Then the operation of partial complex conjugation (we will denote it by $\Psi^{\star 2}$) can be illustrated as follows

$$((\alpha e_1 + \beta e_2) \otimes (\gamma e_1 + \delta e_2))^{\star 2} \equiv (\alpha e_1 + \beta e_2) \otimes (\gamma e_1 + \delta e_2)^\star \equiv (\alpha e_1 + \beta e_2) \otimes (\gamma^\star e_1 + \delta^\star e_2), \tag{9}$$

where the standard basis e_1, e_2 in $\mathcal{C}^2$ was used in the transposition of corresponding projector. Note that the operation of partial conjugation is defined *only* for product vectors.

IV. INSEPARABLE STATES WITH POSITIVE PARTIAL TRANSPOSITION

A. 3×3 *system* .- Consider the Hilbert space $\mathcal{H} = \mathcal{C}^3 \otimes \mathcal{C}^3$. Let $P_\phi \equiv |\phi><\phi|$ and let $\{e_i\}$, i=1, 2, 3 stand for standard basis in $\mathcal{C}^3$. Then we define projector

$$Q \equiv I \otimes I - (\sum_{i=1}^{3} P_{e_i} \otimes P_{e_i} + P_{e_3} \otimes P_{e_1}) \tag{10}$$

and vectors

$$\Psi \equiv \frac{1}{\sqrt{3}}(e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3), \tag{11}$$

$$\Phi_a \equiv e_3 \otimes (\sqrt{\frac{1+a}{2}} e_1 + \sqrt{\frac{1-a}{2}} e_3), 0 \leq a \leq 1 \tag{12}$$

Now we define the following state

$$\varrho_{insep} \equiv \frac{3}{8} P_\Psi + \frac{1}{8} Q. \tag{13}$$

This state is inseparable as its partial transposition possesses a negative eigenvalue $\lambda = \frac{1-\sqrt{5}}{2}$ belonging to the eigenvector $\frac{2}{5+\sqrt{5}}(e_1 \otimes e_3 + \frac{-1-\sqrt{5}}{2} e_3 \otimes e_1)$. Here inseparability comes from highly entangled pure state P_Ψ. On the other hand the state P_{Φ_a} corresponding to the vector (12) is evidently separable. Below we will see that it is possible to mix the states ϱ_{insep} and P_{Φ_a} in such a way that the resulting state will have partial transposition positive being nevertheless inseparable. For this purpose consider the following state

$$\varrho_a = \frac{8a}{8a+1} \varrho_{insep} + \frac{1}{8a+1} P_{\Psi_a}. \tag{14}$$

Its matrix and the matrix of its partial transposition are of the form

$$\varrho_a = \frac{1}{8a+1} \begin{bmatrix} a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+a}{2} & 0 & \frac{\sqrt{1-a^2}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ a & 0 & 0 & 0 & a & 0 & \frac{\sqrt{1-a^2}}{2} & 0 & \frac{1+a}{2} \end{bmatrix}, \quad \varrho_a^{T_2} = \frac{1}{8a+1} \begin{bmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & a & 0 & 0 \\ 0 & a & 0 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & 0 & a & 0 \\ 0 & 0 & a & 0 & 0 & 0 & \frac{1+a}{2} & 0 & \frac{\sqrt{1-a^2}}{2} \\ 0 & 0 & 0 & 0 & 0 & a & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{1-a^2}}{2} & 0 & \frac{1+a}{2} \end{bmatrix}.$$

It is easy to show that $\varrho_a^{T_2}$ is positive. Indeed it suffices only to single out the state $I \otimes U P_{\Phi_a} I \otimes U^\dagger$ as a component of convex combination where

$$U = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{15}$$

and then check that the remaining operator in the combination is positive. Thus $\varrho_a^{T_2}$ is a legitimate state. Now we will show that it is inseparable. Then, as the operation of partial transposition preserves separability, we will have two "dual" sets of inseparable mixtures with positive partial transposition. Let us find all product (unnormalised for convenience)

vectors belonging to the range of $\varrho_a^{T_2}$. We will adopt here the horizontal notation with basis ordered in the following way $e_1 \otimes e_1$, $e_1 \otimes e_2$, $e_1 \otimes e_3$, $e_2 \otimes e_1$, $e_2 \otimes e_2$... and so on. Assume, in addition, that $a \neq 0, 1$. Then any vector belonging to the range of $\varrho_a^{T_2}$ can be presented as

$$u = (A, B, C; B, D, E; C + F, E, xF), \;\; A, B, C, D, E, F \in \mathcal{C}, \tag{16}$$

with nonzero $x = \sqrt{\frac{1+a}{1-a}}$. On the other hand if u is to be positive, it must be of the form

$$u_{prod} = (r, s, t) \otimes (\tilde{A}, \tilde{B}, \tilde{C}) \equiv (r(\tilde{A}, \tilde{B}, \tilde{C}); s(\tilde{A}, \tilde{B}, \tilde{C}); t(\tilde{A}, \tilde{B}, \tilde{C})), \;\; r, s, t, \tilde{A}, \tilde{B}, \tilde{C} \in \mathcal{C}. \tag{17}$$

Let us now consider the following cases:

i) $rs \neq 0$, then without loss generality we can put $r = 1$ and $\tilde{A} = A$, $\tilde{B} = B$, $\tilde{C} = C$. Comparison with (16) gives us in turn: $B = sA$; $E = sC$, $E = tsA \Rightarrow C = tA$ (hence $A \neq 0$ or u_{prod} vanishes); $xF = tC = t^2A$, $C + F = tA$ with $C = tA \Rightarrow F = 0$; $xF = tC = t^2A$ in the presence of vanishing F and non vanishing $A \Rightarrow t = 0$. Thus we obtain the states

$$u_1 = A(1, s, 0) \otimes (1, s, 0), \;\; A, s \in \mathcal{C}. \tag{18}$$

ii) $r = 0$. Then we have

$$u_{prod} = (0, 0, 0; s(\tilde{A}, \tilde{B}, \tilde{C}); t(\tilde{A}, \tilde{B}, \tilde{C})), \;\; s, t, \tilde{A}, \tilde{B}, \tilde{C} \in \mathcal{C}. \tag{19}$$

On the other hand one gets

$$u_{prod} = (0, 0, 0; 0, D, E; F, E, xF), \;\; D, E, F \in \mathcal{C}. \tag{20}$$

Now either $s = 0$ and then, according to (20) $E = 0$ gives us

$$u_2 = F(0, 0, 1) \otimes (1, 0, x), \;\; F \in \mathcal{C} \tag{21}$$

or $s \neq 0$. In the last case we can put $s = 1$. Consequently it is possible that $t = 0$ and then we get via conditions $F = 0$, $E = 0$ another product state

$$u_3 = D(0, 1, 0) \otimes (0, 1, 0), \;\; D \in \mathcal{C}. \tag{22}$$

For the case $t \neq 0$ we get from (19), (20) $\tilde{A} = 0 \Rightarrow F = 0 \Rightarrow E = 0 \Rightarrow D = 0$. Hence the only product vector with non vanishing t is trivial zero vector.

iii) $r \neq 0$, $s = 0$. As in the case of (i) we can put $r = 1$ and $\tilde{A} = A$, $\tilde{B} = B$, $\tilde{C} = C$. Then we have $B = E = D = 0$ which leads to the equality

$$(A, 0, C; 0, 0, 0; t(A, 0, C)) = (A, 0, C; 0, 0, 0; C + F, 0, xF). \tag{23}$$

Then for $t = 0$ we get $C = F = 0$ and

$$u_4 = A(1, 0, 0) \otimes (1, 0, 0),\ \ A \in \mathcal{C}, \tag{24}$$

or, provided that $t \neq 0$, $xF = tC$, $C + F = tA \Rightarrow A = (t^{-1} + x^{-1})C$ and then

$$u_5 = C(1, 0, t) \otimes (t^{-1} + x^{-1}, 0, 1),\ \ C, t \in \mathcal{C}, t \neq 0. \tag{25}$$

All partial complex conjugations of vectors (18), (21), (22), (24), (25) are

$$\begin{aligned}
u_1^{\star 2} &= A(1, s, 0) \otimes (1, s^\star, 0), A, s \in \mathcal{C}, s \neq 0,\\
u_2^{\star 2} &= F(0, 0, 1) \otimes (1, 0, x), F \in \mathcal{C},\\
u_3^{\star 2} &= D(0, 1, 0) \otimes (0, 1, 0), D \in \mathcal{C},\\
u_4^{\star 2} &= A(1, 0, 0) \otimes (1, 0, 0), A \in \mathcal{C},\\
u_5^{\star 2} &= C(1, 0, t) \otimes ((t^\star)^{-1} + x^{-1}, 0, 1), C, t \in \mathcal{C}, t \neq 0.
\end{aligned} \tag{26}$$

It is easy to see that the above vectors can not span the range of ϱ_a as they are orthogonal to the vector

$$\tilde{u} = (0, 0, 1) \otimes (0, 1, 0). \tag{27}$$

which belongs just to $Ran \varrho_a$. Hence, for any $a \neq 0, 1$, the state $\varrho_a^{T_2}$ violates the condition due to the second statement of the theorem. Thus the state is inseparable together with the "dual", original state ϱ_a. In the latter the state P_{Φ_a} masks the inseparability due to ϱ_{insep}, making it "invisible" to the partial transposition criterion, but does not destroy it.

It is interesting to see the limit behaviour of the state ϱ_a. In the case of $a = 0$ we get the separable state with the symmetric representation

$$\varrho_0 \equiv \frac{3}{9} P_\Psi + \frac{1}{9}(I \otimes I - \sum_{i=1}^{3} P_{e_i} \otimes P_{e_i}) = \frac{1}{9} \int_0^{2\pi} P_{\Phi(\phi)} \otimes P_{\Phi(\phi)} \frac{d\phi}{2\pi}, \tag{28}$$

where projectors $P_{\Phi(\phi)}$ correspond to the vectors $\Phi(\phi) = 1/\sqrt{3}(1, e^{i\phi}, e^{-2i\phi})$.

Note that the integral representation (28) is not unique. The representation of the "dual" state $\varrho_a^{T_2}$ is obtained by complex conjugation of projectors acting on the second space. For the case $a = 0$ we get simply the product state P_{Φ_1} (c.f. (12)). Taking the parameter a arbitrarily close to 0, we obtain almost product pure states P_{Φ_a} being nevertheless separable. The situation is, in a sense, analogical to the case of the states introduced in [12]. The inseparability of the latter was also determined by parametric change of both coherences and probabilities involved in the state.

B. 2×4 *system* .- Here we will use the vectors

$$\Psi_i = \frac{1}{\sqrt{2}}(e_1 \otimes e_i + e_2 \otimes e_{i+1}), \quad i = 1, 2, 3, \tag{29}$$

$$\Phi_b \equiv e_2 \otimes (\sqrt{\frac{1+b}{2}} e_1 + \sqrt{\frac{1-b}{2}} e_3), \ 0 \leq b \leq 1. \tag{30}$$

Then we can construct the following state

$$\sigma_{insep} = \frac{2}{7} \sum_{i=1}^{3} P_{\Psi_i} + \frac{1}{7} P_{e_1 \otimes e_4}, \tag{31}$$

which is inseparable (it can be easily verified like in the state σ_{insep} using partial transposition criterion). Now the states of our interest are of the form

$$\sigma_b = \frac{7b}{7b+1} \sigma_{insep} + \frac{1}{7b+1} P_{\Phi_b}. \tag{32}$$

The corresponding matrices are [3]

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[3]The example of pair of matrices of such a type treated, however, as operators on $\mathcal{C}^4 \oplus \mathcal{C}^4$ together with similar analysis of their ranges has been considered in [9] in the context of positive maps.

$$\sigma_b = \frac{1}{7b+1}\begin{bmatrix} b & 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1+b}{2} & 0 & 0 & \frac{\sqrt{1-b^2}}{2} \\ b & 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & b & 0 & \frac{\sqrt{1-b^2}}{2} & 0 & 0 & \frac{1+b}{2} \end{bmatrix}, \quad \sigma_b^{T_2} = \frac{1}{7b+1}\begin{bmatrix} b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & b & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 & b & 0 \\ 0 & b & 0 & 0 & \frac{1+b}{2} & 0 & 0 & \frac{\sqrt{1-b^2}}{2} \\ 0 & 0 & b & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{1-b^2}}{2} & 0 & 0 & \frac{1+b}{2} \end{bmatrix} \tag{33}$$

It is easy to see that the state $\sigma_b^{T_2}$ is positive as

$$\sigma_b^{T_2} = I \otimes U \sigma_b I \otimes U^\dagger, \tag{34}$$

with

$$U = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{35}$$

Consequently, for $y = \sqrt{\frac{1-b}{1+b}} \neq 0, 1$ we get, analogically as in the part A of the section, the partial complex conjugations of all possible product vectors $v_i \in Ran\sigma_b$

$$v_1^{\star 2} = C(1,s) \otimes ((s^\star)^2, s^\star, 1, (s^\star)^{-1}(1 + y(s^\star)^3)), \tag{36}$$

$C, s \in \mathcal{C}, s \neq 0,$

$$v_2^{\star 2} = F(0,1) \otimes (1,0,0,y), \;\; F \in \mathcal{C}$$

$$v_3^{\star 2} = D(1,0) \otimes (0,0,0,1), \;\; D \in \mathcal{C}. \tag{37}$$

On the other hand, any vectors from the range of $\sigma_a^{T_2}$ can be written in our notation as

$$w = (A', B', C', D'; B' + yE', C', D', E'), \;\; A', B', C', D', E' \in \mathcal{C}. \tag{38}$$

Let us check now whether the vectors (37) can be written in the above form. For the $v_1^{\star 2}$, assuming that it is nontrivial one ($C \neq 0$) and at the same time is of the from (38), taking into account the coefficient C' we obtain

$$s^\star = s^{-1}. \tag{39}$$

On the other hand, considering B', E' and $B' + yE'$, we have

$$s^\star + ys(s^\star)^{-1}(1 + y(s^\star)^3) = s(s^\star)^2. \tag{40}$$

Finally taking into account D' we obtain

$$(s^\star)^{-1}(1 + y(s^\star)^3) = s. \tag{41}$$

Combining all the three equations above we find that $ys^2 = 0$, $s \neq 0$ which contradicts the fact that y does not vanish. For $v_2^{\star 2}$ we obtain $yE' = D$ and at the same time $E' = yD$, which is impossible for $y \neq 0, 1$ unless $E' = D = 0$ trivialising then the vector $v_2^{\star 2}$. For the vector $v_3^{\star 2}$ we get immediately that it must hold $D = 0$. It leads to the conclusion that *none* of vectors $v_i^{\star 2}$ belongs to the $Ran\sigma_b^{T_2}$ apart from the trivial zero one. Thus for any $b \neq 0, 1$ the state σ_b violates our criterion from the theorem (statement (ii)) being then inseparable together with its "dual" counterpart $\sigma_b^{T_2}$. Here again the limit cases correspond to separable states. Namely we have:

$$\sigma_0 = \frac{2}{8} P_{\Psi_i} + \frac{1}{8}(P_{e_1 \otimes e_4} + P_{e_4 \otimes e_1}) = \frac{1}{8} \int_0^{2\pi} P_{\psi(\phi)} \otimes Q_{\Psi(\phi)} \frac{d\phi}{2\pi} \tag{42}$$

where $\psi(\phi) = 1/\sqrt{2}(1, e^{i\phi})$ and $\Psi(\phi) = 1/2(1, e^{-i\phi}, e^{-i2\phi}, e^{-i3\phi})$. Putting b=1 we obtain again the separable state state P_{Ψ_1}.

Thus we have provided the families of inseparable states with positive partial transposition. It is natural to ask how they are related to the necessary and sufficient separability condition given in terms of positive maps ([3] , see Introduction). Clearly it follows that, in the presence of inseparability of states ϱ_a, σ_b, there must exist positive maps $\Lambda_a : \mathcal{B}(\mathcal{C}^3) \to \mathcal{B}(\mathcal{C}^3)$ and $\Lambda_b : \mathcal{B}(\mathcal{C}^4) \to \mathcal{B}(\mathcal{C}^2)$ such that the operators $I \otimes \Lambda_a \varrho_a$ and $I \otimes \Lambda_b \sigma_b$ are not positive i.e.

each of them possesses at least one negative eigenvalue. It is easy to see that the maps Λ_a, Λ_b can not be of the form

$$\Lambda = \Lambda_1^{CP} + \Lambda_2^{CP} T, \tag{43}$$

where Λ_i^{CP} are completely positive maps and T is a transposition [3]. However the nature of the positive maps which are not of the form (43) is not known yet and finding the maps Λ_a, Λ_b revealing the inseparability of the states ϱ_a, σ_b may be difficult.

V. CONCLUSION

We have pointed out that any separable state can be written as a convex combination of only N pure product states ($N \leq (dim\mathcal{H})^2$). We have provided a new necessary condition for separability of quantum states in terms of range of density matrices. For any separable state it must be possible to span its range by system of such product vectors that their counterparts obtained by partial complex conjugation span range of partial transposition of the state. It is interesting to see that the above criterion sometimes does not reveal inseparability in cases where the partial transposition one works (it can be see for the case Werner [11] 2×2 states) but it happens to be efficient where the latter fails. Thus, both the criteria are, in general, *independent* for mixed states, although one can easily verify (via Schmidt decomposition) their equivalence for pure states. One could suppose that, taken jointly, they can constitute the new necessary and sufficient condition of separability in higher dimensions. However it is not the case [13]: one can take the states $(1-\epsilon)\varrho_a + \epsilon I/9$, $a \neq 0, 1$, $((1-\epsilon)\sigma_b + \epsilon I/4$, $b \neq 0, 1)$. Those states will obviously satisfy the partial transposition criterion and they would also satisfy the present one as the latter is useful only for the states with range essentially less then $\mathcal{H}$. Now, as the set of separable states is closed, taking sufficiently small ϵ one can ensure that the new states remain inseparable. And this fact is present despite they satisfy both mentioned criteria.

The present criterion allowed us to provide examples of states of a new kind, where the entanglement is masked in a specific way by a classical admixture. In this context an

interesting problem arises whether it is possible to distill such an entanglement using local operations and classical communication.

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